Mathematical Finance
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Assignment 7—solutions

We fix throughout a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which we are given a filtration $\mathbb{F}$, unless otherwise stated.

A large financial market

We take here as a probability space $\Omega := [0,1]$, $\mathcal{F}$ the Borel–σ-algebra on $[0,1]$, and as probability measure $\mathbb{P}$ the Lebesgue measure on $[0,1]$. We consider then a financial market with time-horizon 1, and with countably many risky assets with (discounted) prices $(S^n)_{n\in\mathbb{N}}$ which are given for any $n \in \mathbb{N}$ by

$$S^n_t := 0,\ t \in [0,1),\ S^n_1(x) := \begin{cases} -x^{-1/2},\ \text{if } x \in [0,\varepsilon_n), \\ (1-x)^{-\frac{1}{n+1}},\ \text{if } x \in [\varepsilon_n, 1], \end{cases}$$

where the sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ takes values in $(0,1)$, and converges to 0 as n goes to $+\infty$. We take for $\mathbb{F}$ the natural filtration generated by $(S^n)_{n\in\mathbb{N}}$.

1) Show that it is possible to choose the sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $\mathbb{E}^{\mathbb{P}}[S^n_1] = 1$ for all $n \in \mathbb{N}$.

   We have for any $n \in \mathbb{N}$

   $$\mathbb{E}^{\mathbb{P}}[S^n_1] = -\int_0^{\varepsilon_n} \frac{\mathrm{d}x}{\sqrt{x}} + \int_{\varepsilon_n}^1 \frac{\mathrm{d}x}{(1-x)^{\frac{1}{n+1}}} = -2\sqrt{\varepsilon_n} + \frac{n+1}{n}(1-\varepsilon_n)^{\frac{n}{n+1}},$$

   so that

   $$\mathbb{E}^{\mathbb{P}}[S^n_1] = 1 \Longleftrightarrow 1 + 2\sqrt{\varepsilon_n} = \frac{n+1}{n}(1-\varepsilon_n)^{\frac{n}{n+1}}.$$

   Define then for any $n \in \mathbb{N}$ the map $f_n : (0,1) \longrightarrow \mathbb{R}$ by

   $$f_n(x) := 1 + 2\sqrt{x} - \frac{n+1}{n}(1-x)^{\frac{n}{n+1}}.$$

   We have

   $$f'_n(x) = \frac{1}{\sqrt{x}} + (1-x)^{-\frac{1}{n+1}}, x \in (0,1),$$

   so that f_n is increasing for any $n \in \mathbb{N}$, with $\lim_{x\to 0+} f_n(x) = -1/n$ and $\lim_{x\to 1-} f_n(x) = 3$, meaning that there is a unique solution $\varepsilon_n \in (0,1)$ to the equation $f_n(x) = 0$. Moreover, as n goes to $+\infty$, it is clear that we must have that ε_n goes to 0, otherwise the equality

   $$1 + 2\sqrt{\varepsilon_n} = \frac{n+1}{n}(1-\varepsilon_n)^{\frac{n}{n+1}},$$

   could not be satisfied, since its right-hand side must go to 1 as n goes to $+\infty$.

2) We now want to prove that $\mathbb{P}$ is a separating measure for this market. Show that it is enough for this to prove that for any $n \in \mathbb{N}$ and any sequence $(c_k)_{k\in\{0,\dots,n\}}$ such that $\sum_{k=0}^n c_k S^k_1$ is bounded from below, we have

   $$\mathbb{E}^{\mathbb{P}}\left[\sum_{k=0}^n c_k S^k_1\right] \leq 0,$$

   and deduce that $\mathbb{P}$ is indeed a separating measure.

   In this market, the terminal wealth at time 1 of an admissible portfolio takes the form

   $$X_1 = \sum_{n\in\mathbb{N}} c_n S^n_1,$$

for some sequence $(c_n)_{n\in\mathbb{N}}$ such that X_1 is bounded from below. Indeed, the asset prices are equal to 0 before the terminal time 1, so admissible portfolio processes must have this form. We want to prove that for any such X_1, $\mathbb{E}^{\mathbb{P}}[X_1]\leq 0$. By Fatou's lemma (recall that the sums appearing are bounded from below), it is enough to prove that for any $n\in\mathbb{N}$

$$\mathbb{E}^{\mathbb{P}}\Bigg[\underbrace{\sum_{k=0}^{n} c_k S_1^k}_{=:X_1^n}\Bigg]\leq 0.$$

We have

$$\mathbb{E}^{\mathbb{P}}[X_1^n]=\sum_{\{k\in\{0,\dots,n\}:c_k>0\}} c_k\mathbb{E}^{\mathbb{P}}[S_1^k]+\sum_{\{k\in\{0,\dots,n\}:c_k<0\}} c_k\mathbb{E}^{\mathbb{P}}[S_1^k]=\sum_{\{k\in\{0,\dots,n\}:c_k>0\}} c_k-\sum_{\{k\in\{0,\dots,n\}:c_k<0\}} |c_k|$$

and since, for $x\in[0,\varepsilon_n]$, we have

$$X_1^n(x)=-\frac{1}{\sqrt{x}}\Bigg(\sum_{\{k\in\{0,\dots,n\}:c_k>0\}} c_k-\sum_{\{k\in\{0,\dots,n\}:c_k<0\}} |c_k|\Bigg)=-\frac{1}{\sqrt{x}}\mathbb{E}^{\mathbb{P}}[X_1^n],$$

the only way X_1^n can remain bounded from below for any $n\in\mathbb{N}$ is if $\mathbb{E}^{\mathbb{P}}[X_1^n]\leq 0$.

3) Prove that there cannot exist an equivalent σ-martingale measure on this market, and comment.

If $\mathbb{Q}$ is an equivalent σ-martingale measure, then for any $n\in\mathbb{N}$, S^n must be an $(\mathbb{F},\mathbb{Q})$–σ-martingale. This means that there is a sequence $(D_k)_{k\in\mathbb{N}}$ of $\mathbb{F}$-predictable sets whose union is $\Omega\times[0,1]$ and such that for any $k\in\mathbb{N}$, the process $Y^{k,n}:=\int_0^{\cdot}\mathbf{1}_{D_k}(s)\mathrm{d}S_s^n=\mathbf{1}_{\{\cdot=1\}}\mathbf{1}_{D_k}(1)S_1^n$, is a $\mathbb{Q}$–uniformly integrable $(\mathbb{F},\mathbb{Q})$-martingale. This implies in particular that $\mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{D_k}(1)S_1^n]=0$, which is equivalent to

$$\int_0^{\varepsilon_n}\frac{1}{\sqrt{x}}\mathbf{1}_{D_k}(1,x)\mathrm{d}\mathbb{Q}(x)=\int_{\varepsilon_n}^1\frac{1}{(1-x)^{\frac{1}{n+1}}}\mathbf{1}_{D_k}(1,x)\mathrm{d}\mathbb{Q}(x).$$

The term to the left-hand side must go to 0 as ε go to 0 since $\mathbb{Q}$ is equivalent to Lebesgue measure. However, we have

$$\int_{\varepsilon_n}^1\frac{1}{(1-x)^{\frac{1}{n+1}}}\mathbf{1}_{D_k}(1,x)\mathrm{d}\mathbb{Q}(x)\geq\int_{\varepsilon_n}^1\mathbf{1}_{D_k}(1,x)\mathrm{d}\mathbb{Q}(x).$$

This would thus imply that $\int_0^1\mathbf{1}_{D_k}(1,x)\mathrm{d}\mathbb{Q}(x)=0$, and thus letting k go to $+\infty$ by dominated convergence that $\mathbb{Q}[[0,1]]=0$, which contradicts the fact that $\mathbb{P}$ and $\mathbb{Q}$ are equivalent.

This means that for markets with infinitely many assets, the existence of a separating measure no longer implies the existence of a σ-martingale measure, and that the form of the first FTAP must be modified. For more information, you can see for instance Cuchiero, Klein, and Teichmann [1].

On separating measures

Consider a financial market where discounted prices are given by $S:=(S^1,\dots,S_t^d)^\top_{t\in[0,T]}$ which is a d-dimensional $(\mathbb{F},\mathbb{P})$–semi-martingale and let $\mathbb{Q}$ be a measure equivalent to $\mathbb{P}$ on $\mathcal{F}_T$

1) Assume that $\mathcal{F}_0$ is trivial and that $\mathbb{Q}$ is a separating measure for S. Show that if S is $(\mathbb{F},\mathbb{P})$–locally bounded, then $\mathbb{Q}$ is an equivalent local martingale measure for S.

First, assume that S is bounded. Note that then every simple strategy is admissible. Moreover, S is a $\mathbb{Q}$–uniformly integrable $(\mathbb{F},\mathbb{Q})$-martingale if and only if $\mathbb{E}^{\mathbb{Q}}[S_\tau-S_0]=0$ for all $\mathbb{F}$–stopping times

τ taking values in $[0,T]$. So let τ be such an arbitrary $\mathbb{F}$–stopping time, and consider the simple strategies $\xi^{\pm} := \pm\mathbf{1}_{]\!]0,\tau]\!]}$. Using that $\mathbb{Q}$ is an equivalent separating measure for S then gives

$$0 \geq \mathbb{E}^{\mathbb{Q}}\left[\int_0^T \xi_s^{\pm}\mathrm{dS}_s\right] = \pm\mathbb{E}^{\mathbb{Q}}[S_\tau - S_0]. \tag{0.1}$$

If S is $(\mathbb{F},\mathbb{P})$–locally bounded, then there exists an increasing sequence of $\mathbb{F}$–stopping times $(\sigma_n)_{n\in\mathbb{N}}$ taking values in $[0,T]$ with $\lim_{n\to\infty}\mathbb{P}[\sigma_n = T] = 1$ such that S^{σ_n} is bounded for all $n\in\mathbb{N}$. It suffices to show that for each $n\in\mathbb{N}$, S^{σ_n} is a $\mathbb{Q}$–uniformly integrable $(\mathbb{F},\mathbb{Q})$-martingale. To this end, fix $n\in\mathbb{N}$. It suffices to show that for each $\mathbb{F}$–stopping time τ with $\tau\leq\sigma_n$, $\mathbb{P}$–a.s., $\mathbb{E}^{\mathbb{Q}}[S_\tau - S_0] = 0$. So let τ be such an $\mathbb{F}$–stopping time, and consider as above the simple strategies $\xi^{\pm} := \pm\mathbf{1}_{]\!]0,\tau]\!]}$. Then both strategies are admissible since S is bounded on $[\![0,\sigma_n]\!]$ and $\tau\leq\sigma_n$ $\mathbb{P}$–a.s., and the same argument as in the first step gives $\mathbb{E}^{\mathbb{Q}}[S_\tau - S_0] = 0$.

2) Assume that $\mathbb{Q}$ is an equivalent $(\mathbb{F},\mathbb{Q})$–σ-martingale measure for S. Show that it is also an equivalent separating measure.

By assumption, there exist a strictly positive predictable process $\psi = (\psi_t)_{t\in[0,T]}$, an $\mathbb{R}^d$-valued $(\mathbb{F},\mathbb{Q})$–local martingale M and a $\mathbb{R}^d$-valued $\mathcal{F}_0$-measurable random vector S_0 such that $S = S_0 + \int_0^\cdot \psi_s\cdot\mathrm{d}M_s$. Let ξ be an $\mathbb{F}$-predictable process such that $\int_0^\cdot \xi_s\cdot\mathrm{d}S_s$ is bounded from below. Then by the associativity of the stochastic integral, $\int_0^\cdot \xi_s\cdot\mathrm{d}S_s = \int_0^\cdot \xi_s\psi_s\cdot\mathrm{d}M_s$. Moreover, since $\int_0^\cdot \xi_s\psi_s\cdot\mathrm{d}M_s$ is uniformly bounded from below by admissibility, it is an $(\mathbb{F},\mathbb{Q})$–local martingale by the Ansel–Stricker lemma. By Fatou's lemma, it is then also an $(\mathbb{F},\mathbb{Q})$–super-martingale, and hence

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \xi_s\cdot\mathrm{d}S_s\right] \leq 0. \tag{0.2}$$

3) Now assume that $d=1$, that $(\mathcal{F}_t)_{t\in[0,T]}$ is the natural ($\mathbb{P}$-completed) filtration of S and that the process $S = (S_t)_{t\in[0,T]}$ is of the form

$$S_t = \begin{cases} 0, \text{ if } 0\leq t<T, \\ X, \text{ if } t=T, \end{cases}$$

where X is normally distributed with mean $\mu\neq 0$ and variance $\sigma^2>0$ under $\mathbb{P}$. Show that in this case, the class $\mathcal{M}_{\mathrm{sep}}(S,\mathbb{F},\mathbb{P})$ of equivalent separating measures for S is strictly bigger than $\mathcal{M}_\sigma(S,\mathbb{F},\mathbb{P})$.

Let $\xi\in\mathcal{L}^1(S,\mathbb{F},\mathbb{P})$ be arbitrary. Then

$$\int_0^T \xi_s\cdot\mathrm{d}S_s = \lim_{t\uparrow T}\int_0^t \xi_s\cdot\mathrm{d}S_s + \xi_T X = \xi_T X. \tag{0.3}$$

Since ξ is $\mathbb{F}$-predictable where $\mathbb{F}$ is the natural filtration of S, ξ_T is $\mathcal{F}_{T-}$-measurable, and therefore deterministic. Since X has unbounded support, $\int_0^T \xi_s\cdot\mathrm{d}S_s$ is bounded from below if and only if $\xi_T = 0$. Thus, we may conclude that $\int_0^T \xi_s\cdot\mathrm{d}S_s = 0$ for all admissible ξ.

Therefore the condition

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \xi_s\cdot\mathrm{d}S_s\right] \leq 0, \text{ for all admissible } \xi,$$

is trivially satisfied for each probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $\mathcal{F}_T$. In particular, $\mathbb{P}$ itself is a separating measure for S.

Finally if $\mathbb{Q}$ is an equivalent probability measure, by 1) (whose results remain unchanged by an equivalent change of measure), $M = (M_t)_{t\in[0,T]}$ is a $\mathbb{Q}$-martingale null at 0 for the filtration $(\mathcal{F}_t)_{t\in[0,T]}$

if and only if M_T is $\sigma(X)$-measurable, $\mathbb{Q}$-integrable with mean 0 and $M_t = 0$ for all $t \in [0,T)$. Moreover, if $\psi \in \mathcal{L}^1(M, \mathbb{F}, \mathbb{Q})$, then as M is constant and equal to 0 on $[0,T)$

$$\int_0^t \psi_s \cdot \mathrm{d}M_s = \begin{cases} 0, \ \textbf{for } t < T, \\ \psi_T M_T, \ \textbf{for } t = T. \end{cases} \tag{0.4}$$

Note that as ψ_T is constant, $\int_0^\cdot \psi_s \cdot \mathrm{d}M_s$ is a true $(\mathbb{F}, \mathbb{Q})$-martingale, and thus $\mathbb{Q}$ is an equivalent σ-martingale measure for S if and only if it is an equivalent martingale measure. Since $\mathbb{E}^{\mathbb{P}}[S_T] = \mu \neq 0$, $\mathbb{P}$ is not a martingale measure and hence also not a σ-martingale measure.

Stop–loss start–gain strategy

Let the financial market on $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, $T < \infty$, be described by a reference asset $S^0 = 1$ and one risky asset S being a geometric Brownian motion, i.e.

$$\mathrm{d}S_t = S_t\big(\mu \mathrm{d}t + \sigma \mathrm{d}W_t\big), \ S_0 = s_0 > 0, \tag{0.5}$$

for some given constants $\mu \in \mathbb{R}$, $\sigma > 0$.

Fix $K > 0$. We start with one share if $S_0 > K$ and with no share if $S_0 \leq K$. Whenever the stock price falls below K (or equals K), the share is sold, and whenever the price returns to a level strictly above K, one share is bought again. Thus, the amount held in the reference asset is given by $\delta_t = -K\mathbf{1}_{\{S_t > K\}}$, $t \in [0,T]$, and the amount held in the risky asset is given by $\Delta_t = \mathbf{1}_{\{S_t>K\}}$, $t \in [0,T]$.

1) Verify that the geometric Brownian motion S satisfying (0.5) has the expression

$$S_t = s_0 \exp\big(\sigma W_t + (\mu - \sigma^2/2)t\big), \ t \in [0,T].$$

Apply Itô's formula to $S_t = s_0 \exp(\sigma W_t + (\mu - \frac{1}{2}\sigma^2)t)$ to see that S satisfies the desired dynamics. Uniqueness is standard.

2) Show that for each $t \in (0,T]$, it holds that

$$\mathbb{P}[S_t > K] > 0, \ \text{and } \mathbb{P}[S_t < K] > 0.$$

Note that $\{S_t > K\} = \big\{W_t > \frac{1}{\sigma}(\log(K/s_0) - (\mu - \frac{1}{2}\sigma^2)t)\big\}$. Since under the measure $\mathbb{P}$, the random variable W_t has a normal distribution, we get

$$\mathbb{P}[S_t > K] = \mathbb{P}\bigg[W_t > \frac{1}{\sigma}(\log(K/s_0) - (\mu - \sigma^2/2)t)\bigg] > 0.$$

Similarly we have $\mathbb{P}[S_t < K] > 0$.

3) Let $L^K(S)$ be the local time of S at K defined as in the lecture notes. Show that $\mathbb{P}[L^K_t(S) > 0] > 0$ holds for all $t \in (0,T]$.

Hint: Recall that by Girsanov's theorem, there exists a measure $\mathbb{Q}$ which is equivalent to $\mathbb{P}$ on $\mathcal{F}_T$ and such that $(S_t)_{t\in[0,T]}$ is an $(\mathbb{F}, \mathbb{Q})$–martingale. You can take the $\mathbb{Q}$-expectation of $(S_t - K)^+$ and apply Jensen's inequality to get the desired result. Tanaka's formula will be very helpful. You may also use the fact that if S is a continuous martingale and H is a bounded $\mathbb{F}$-predictable process, then the stochastic integral $\int_0^\cdot H \mathrm{d}S$ is also a continuous martingale.

Let $\mathbb{Q}$ be an equivalent measure on $\mathcal{F}_T$ for S such that S is an $(\mathbb{F}, \mathbb{Q})$-martingale. By Tanaka's formula

$$(S_t - K)^+ = (S_0 - K)^+ + \int_0^t \mathbf{1}_{\{S_s > K\}} \mathrm{d}S_s + \frac{1}{2}L^K_t(S), \ t \in [0,T],$$

and also note that with S being an $(\mathbb{F}, \mathbb{Q})$-martingale, the stochastic integral $\int_0^\cdot \mathbf{1}_{\{S_s>K\}} \mathrm{d}S_s$ is also an $(\mathbb{F}, \mathbb{Q})$-martingale. Hence, taking the $\mathbb{Q}$-expectation of both sides of the equation above, we get for any $t \in [0, T]$

$$\mathbb{E}^{\mathbb{Q}}[(S_t - K)^+] - \mathbb{E}^{\mathbb{Q}}[(S_0 - K)^+] = \frac{1}{2}\mathbb{E}^{\mathbb{Q}}[L_t^K(S)].$$

Since $\mathbb{Q}$ is equivalent to $\mathbb{P}$, we can derive from 2) that $\mathbb{Q}[S_t > K] > 0$ and $\mathbb{Q}[S_t < K] > 0$. Consequently, since the function $g(x) := (x-K)^+$ is strictly convex on any interval containing K, Jensen's inequality applied for $\mathbb{E}^{\mathbb{Q}}[g(S_t)]$ is strict and therefore for any $t \in [0, T]$

$$\frac{1}{2}\mathbb{E}^{\mathbb{Q}}[L_t^K(S)] = \mathbb{E}^{\mathbb{Q}}[g(S_t)] - \mathbb{E}^{\mathbb{Q}}[g(S_0)] > g\big(\mathbb{E}^{\mathbb{Q}}[S_t]\big) - g(s_0) = g(s_0) - g(s_0) = 0.$$

It follows that for any $t \in [0, T]$, $\mathbb{Q}[L_t^K(S) > 0] > 0$ and of course also $\mathbb{P}[L_t^K(S) > 0] > 0$.

4) Conclude that the so-called *stop–loss start–gain strategy* (δ, Δ) defined above is not a self-financing strategy.

We first observe that the portfolio value at time $t > 0$ is given for any $t \in [0, T]$ by

$$X_t^{\delta,\Delta} = \delta_t + \Delta_t S_t = -K\mathbf{1}_{\{S_t>K\}} + \mathbf{1}_{\{S_t>K\}} S_t = \max(0, S_t - K) = (S_t - K)^+.$$

By definition, (δ^0, Δ) is self-financing if and only if for any $t > 0$

$$X_t^{\delta,\Delta} = X_0^{\delta,\Delta} + \int_0^t \Delta_s \mathrm{d}S_s. \tag{0.6}$$

Now by Tanaka's formula and noting that $X_0^{\delta,\Delta} = (S_0 - K)^+$, we have

$$X_t^{\delta,\Delta} = (S_t - K)^+ = (S_0 - K)^+ + \int_0^t \mathbf{1}_{\{S_s>K\}} \mathrm{d}S_s + \frac{1}{2}L_t^K(S). \tag{0.7}$$

Thus, we see from the comparison of (0.6) with (0.7) that (δ, Δ) is self-financing if and only if for any $t > 0$, $L_t^K(S)$ is equal to zero $\mathbb{P}$–a.s. But we know from 3) that $L_t^K(S) \geq 0$, $\mathbb{P}$–a.s. and $\mathbb{P}[L_t^K(S) > 0] > 0$, and hence (δ, Δ) is not self-financing.

References

[1] C. Cuchiero, I. Klein, and J. Teichmann. A new perspective on the fundamental theorem of asset pricing for large financial markets. *Theory of Probability & Its Applications*, 60(4):561–579, 2016.